EXISTENCE OF ANTIPARTICLES AS AN INDICATION OF FINITENESS OF NATURE

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Abstract:

It is shown that in a quantum theory based on a Galois field, the famous Dirac's result about antiparticles is generalized such that a particle and its antiparticle are already combined at the level of irreducible representations of the symmetry algebra without assuming the existence of a local covariant equation. We argue that the very existence of antiparticles is a strong indication that nature is described by a finite field rather than by complex numbers.

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1 Motivation

The most striking feature of the modern quantum theory is probably the following. On one hand, this theory describes many experimental data with an unprecedented accuracy. On the other hand, the mathematical substantiation of the theory is rather poor. As a consequence, the issue of infinities is probably the most challenging problem in standard formulation of quantum theory. While in QED and other renormalizable theories this problem can be somehow circumvented, in quantum gravity this is not possible even in the lowest orders of perturbation theory.

Mathematical problems of quantum theory are discussed in a wide literature. For example, in the well known textbook [1] it is explained in details that interacting quantized fields can only be treated as operatorial distributions and hence their product at the same point is not well defined. One of ideas of the string theory is that if a point (a zero-dimensional object) is replaced by a string (a one-dimensional object) then there is hope that infinities will be less singular.

There exists a wide literature aiming to solve the difficulties of the theory by replacing the field of complex numbers by quaternions, p-adic numbers or other constructions. A modern state-of-the-art of the p-adic theory can be found, for example, in Ref. [2]. At present it is not clear how to overcome all the difficulties but

at least from the point of view of the problem of infinities a natural approach is to consider quantum theory over Galois fields (GFQT). Since any Galois field is finite, the problem of infinities in GFQT does not exist in principle and all operators are well defined. The idea of using finite fields in quantum theory has been discussed by several authors (see e.g. Refs. [3, 4]). As stated in Ref. [4], a fundamental theory can be based either on p-adic numbers or finite fields. In that case, a correspondence with the standard theory will take place if the number p in the p-adic theory or as a characteristic of a finite field is rather large.

The authors of Ref. [4] and many other papers argue that fundamental quantum theory cannot be based on mathematics using standard geometrical objects (such as strings etc.) at Planck distances. We believe it is rather obvious that the notions of continuity, differentiability, smooth manifolds etc. are based on our macroscopic experience. For example, the water in the ocean can be described by equations of hydrodynamics but we know that this is only an approximation since matter is discrete. Therefore continuous geometry probably does not describe physics even at distances much greater than the Planck length (also see the discussion below).

In our opinion, an approach based on finite fields is very attractive for solving problems in quantum theory as well as for philosophical and aesthetical reasons. Below we describe some arguments in favor of this opinion.

The key ingredient of standard mathematics is the notions of infinitely small and infinitely large numbers. The notion of infinitely small numbers is based on our everyday experience that any macroscopic object can be divided by 2, 10, 1000 etc. In the view of the existence of elementary particles, the notion of division has a limited applicability. For example, we cannot divide the electron or neutrino by two. Therefore, *if we accept the existence of elementary particles, we should acknowledge that our experience based on standard mathematics is not universal.*

The notion of infinitely large numbers is based on the belief that *in principle* we can operate with any large numbers. In standard mathematics this belief is formalized in terms of axioms (accepted without proof) about infinite sets (e.g. Zorn's lemma or Zermelo's axiom of choice). At the same time, in the spirit of quantum theory, there should be no statements accepted without proof since only those statements have physical significance, which can be experimentally verified, at least in principle.

For example, we cannot verify that $a+b=b+a$ for any numbers a and b. Suppose we wish to verify that 100+200=200+100. In the spirit of quantum theory, it is insufficient to say that 100+200=300 and 200+100=300. To check these relationships, we should describe an experiment where they can be verified. In particular, we should specify whether we have enough resources to represent the numbers 100, 200 and 300. We believe the following observation is very important: although standard mathematics is a part of our everyday life, people typically do not realize that *standard mathematics is implicitly based on the assumption that one can have any desirable amount of resources.*

Suppose that our Universe is finite. This implies that the amount of resources cannot be infinite and it is natural to assume that there exists a number p such that all calculations can be performed only modulo p. In this case, one might consider a quantum theory over a Galois field with the characteristic p. Since any Galois field is finite, the fact that arithmetic in this field is correct can be verified, at least in principle, by using a finite amount of resources.

If one accepts the idea to replace complex numbers by a Galois field, the problem arises what formulation of the standard quantum theory is most convenient for that purpose. A well known historical fact is that originally quantum theory has been proposed in two formalisms which seemed essentially different: the Schroedinger wave formalism and the Heisenberg operator (matrix) formalism. It has been shown later by Born, von Neumann and others that both formalisms are equivalent and, in addition, the path integral formalism has been developed.

In the spirit of the wave or path integral approach one might try to replace classical spacetime by a finite lattice which may even not be a field. In that case the problem arises what the natural 'quantum of spacetime' is and some of physical quantities should necessarily have the field structure. A detailed discussion can be found in Ref. [3] and references therein. In contrast to these approaches, we propose to generalize the standard operator formulation, where quantum systems are described by elements of a projective complex Hilbert spaces and physical quantities are represented by self-adjoint operators in such spaces.

From the point of view of quantum theory, any physical quantity can be discussed only in conjunction with the operator defining this quantity. However, in textbooks on quantum mechanics it is usually not indicated explicitly that the quantity t is a parameter, which has the meaning of time only in the classical limit since there is no operator corresponding to this quantity. Since the 1930's it has been well known [5] that, when quantum mechanics is combined with relativity, there is no operator satisfying all the properties of the spatial position operator. In other words, the coordinate cannot be exactly measured even in situations when exact measurement is allowed by the nonrelativistic uncertainty principle. In the introductory section of the well-known textbook [6] simple arguments are given that for a particle with mass m, the coordinate cannot be measured with the accuracy better than the Compton wave length $\hbar/mc$. Hence, the exact measurement is possible only either in the nonrelativistic limit (when $c \to \infty$) or classical limit (when $\hbar \to 0$). From the point of view of quantum theory, one can discuss if the *coordinates of particles* can be measured with a sufficient accuracy, while the notion of empty spacetime background fully contradicts basic principles of this theory. Indeed, the coordinates of points, which exist only in our imagination are not measurable. In particular, the quantity x in the Lagrangian density $L(x)$ is not measurable. Note that the Lagrangian is only an auxiliary tool for constructing Hilbert spaces and operators and this is all we need to have the maximum possible information in quantum theory. After this construction has been done, one can safely forget about Lagrangian and concentrate

his or her efforts on calculating different observables. As stated in Ref. [6], local quantum fields and Lagrangians are rudimentary notion, which will disappear in the ultimate quantum theory. Analogous ideas were the basis of the Heisenberg S-matrix program.

In view of the above discussion, GFQT could be defined as a theory where

- *Quantum states are represented by elements of a linear projective space over a Galois field and physical quantities are represented by linear operators in that space.*

As noted in Ref. [2] and references therein, in the p-adic theory a problem arises what number fields (if any) are preferable and there should be quantum fluctuations not only of metrics and geometry but also of the number field. Volovich [4] proposed the following number field invariance principle: fundamental physical laws should be invariant under the change of the number field. Analogous questions can be posed in GFQT.

It is well known (see e.g. standard textbooks [7]) that any Galois field can contain only p^n elements where p is prime and n is natural. Moreover, the numbers p and n define the Galois field up to isomorphism. It is natural to require that there should exist a correspondence between any new theory and the old one, i.e. at some conditions both theories should give close predictions. In particular, there should exist a large number of quantum states for which the probabilistic interpretation is valid. Then, as shown in our papers [8, 9], in agreement with Refs. [3, 4], the number p should be very large. Hence, we have to understand whether there exist deep reasons for choosing a particular value of p or it is simply an accident that our Universe has been created with this value. Since we don't know the answer, we accept a simplest version of GFQT, where there exists only one Galois field with the characteristic p, which is a universal constant for our Universe. Then the problem arises what the value of n is. Since there should exist a correspondence between GFQT and the complex version of standard quantum theory, a natural idea is to accept that the principal number field in GFQT is the Galois field analog of complex numbers which is constructed below.

Let $F_p = Z/pZ$ be the residue field modulo p and F_{p^2} be a set of p^2 elements $a + bi$ where $a, b \in F_p$ and i is a formal element such that $i^2 = -1$. The question arises whether F_{p^2} is a field, i.e. one can define all the arithmetic operations except division by zero. The definition of addition, subtraction and multiplication in F_{p^2} is obvious and, by analogy with the field of complex numbers, one could define division as $1/(a + bi) = a/(a^2 + b^2) - ib/(a^2 + b^2)$ if a and b are not equal to zero simultaneously. This definition can be meaningful only if $a^2 + b^2 \neq 0$ in F_p. If a and b are not simultaneously equal to zero, this condition can obviously be reformulated such that -1 should not be a square in F_p (or in terminology of number theory it should not be a quadratic residue). We will not consider the case $p = 2$ and then p is necessarily odd. Then we have two possibilities: the value of $p\,(mod\,4)$ is either 1 or

3. The well known result of number theory is that -1 is a quadratic residue only in the former case and a quadratic nonresidue in the latter one, which implies that the above construction of the field F_{p^2} is correct only if $p = 3\ (mod\, 4)$.

The main idea of establishing the correspondence between GFQT and the standard theory is as follows (see Refs. [8, 9] for a detailed discussion). The first step is to notice that the elements of F_p can be written not only as $0, 1, ...p-1$ but also as $0, \pm 1, ..., \pm(p-1)/2$. Such elements of F_p are called minimal residues [7]. Since the field F_p is cyclic, it is convenient to visually depict its elements by the points of a circumference of the radius $p/2\pi$ on the plane (x, y) such that the distance between neighboring elements of the field is equal to unity, and the elements 0, 1, 2,... are situated on the circumference counterclockwise. At the same time we depict the elements of Z as usual, such that each element $z \in Z$ is depicted by a point with the coordinates $(z, 0)$. In Fig. 1 a part of the circumference near the origin is depicted. Let f be a map from F_p to Z such that $f(a)$ has the same

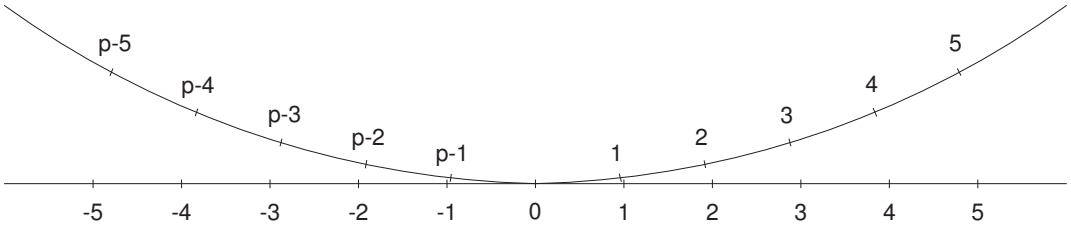

Figure 1: Relation between F_p and the ring of integers

notation in Z as its minimal residue in F_p. Then for elements $a, b \in F_p$ such that $|f(a)|, |f(b)| \ll p$, addition, subtraction and multiplication in F_p and Z are the same, i.e. $f(a \pm b) = f(a) \pm f(b)$ and $f(ab) = f(a)f(b)$.

The second step is to establish a correspondence between Hilbert spaces in standard theory and spaces over a Galois field in GFQT. We first note that Hilbert spaces contain a big redundancy of elements and we do not need to know all of them. With any desired accuracy we can approximate each element $\tilde{x}$ from a Hilbert space H by a finite linear combination $\tilde{x} = \tilde{c}_1\tilde{e}_1 + \tilde{c}_2\tilde{e}_2 + ...\tilde{c}_n\tilde{e}_n$ where $(\tilde{c}_1, \tilde{c}_2, ...\tilde{c}_n)$ are rational complex numbers. In turn, the set of such elements is redundant too. We can use the fact that Hilbert spaces in quantum theory are projective: ψ and $c\psi$ represent the same physical state, which reflects the fact that not the probability itself but the relative probabilities of different measurement outcomes have a physical meaning. Therefore we can multiply both parts of the above equality by a common denominator of the numbers $(\tilde{c}_1, \tilde{c}_2, ...\tilde{c}_n)$. As a result, we can always assume that $\tilde{c}_j = \tilde{a}_j + i\tilde{b}_j$ where $\tilde{a}_j$ and $\tilde{b}_j$ are integers.

Consider now a space over F_{p^2} and let $x = c_1e_1 + c_2e_2 + ...c_ne_n$ be a decomposition of a state x over a basis $(e_1, e_2...)$ in this space. We can formally define a scalar product (e_j, e_k) such that $f((e_j, e_k)) = (\tilde{e}_j, \tilde{e}_k)$. Then the correspondence between the states x and $\tilde{x}$ can be defined such that $c_j = a_j + ib_j$ $(j = 1, 2...)$,

$f(a_j) = \tilde{a}_j$ and $f(b_j) = \tilde{b}_j$. If the numbers in question are much less than p then the standard description and that based on GFQT give close experimental predictions. At the same time, in GFQT a probabilistic interpretation is not universal and is valid only when the numbers in question are much less than p.

The above discussion has a well known historical analogy. For many years people believed that our Earth was flat and infinite, and only after a long period of time they realized that it was finite and had a curvature. It is difficult to notice the curvature when we deal only with distances much less than the radius of the curvature R. Analogously one might think that the set of numbers describing physics has a curvature defined by a very large number p but we do not notice it when we deal only with numbers much less than p.

Since we treat GFQT as a more general theory than the standard one, it is desirable not to postulate that GFQT is based on F_{p^2} (with $p = 3\ (mod\, 4)$) because standard theory is based on complex numbers but vice versa, explain the fact that standard theory is based on complex numbers since GFQT is based on F_{p^2}. Hence, one should find a motivation for the choice of F_{p^2} in GFQT. A possible motivation is discussed in Refs. [9, 10]. The main result of the present paper does not depend on the choice of the number field.

In standard approach to symmetries in quantum theory, the symmetry group is a group of motions of a classical background spacetime. As noted above, in quantum theory the background spacetime does not have a physical meaning. So a question arises whether there exists an alternative for such an approach. As already noted, in the standard approach, the background spacetime and Lagrangian are only auxiliary tools for constructing Hilbert spaces and operators. For calculating observables one needs not representation operators of the symmetry group but representation operators of its Lie algebra, e.g. the Hamiltonian. The representation operators of the group are needed only if it is necessary to calculate macroscopic transformations, e.g. spacetime transformations. In the approximation when classical time and space are good approximate parameters, the Hamiltonian and momentum operators can be interpreted as ones associated with the corresponding translations, but nothing guarantees that this interpretation is always valid (e.g. at the very early stage of the Universe). One might think that this observation is not very significant, since typically symmetry groups are Lie groups and for them in many cases there exits a one-to-one correspondence between representations of the Lie group and its Lie algebra. However, in Galois fields there is no notion of infinitesimal transformations and hence there is no notion of Lie group over a Galois field associated with a given Lie algebra over a Galois field. In the spirit of Dirac's paper [11], we postulate that on quantum level a symmetry means that a system is described by a set of operators, which satisfy certain commutation relations.

In view of these remarks, and by analogy with standard quantum theory it is natural to define the elementary particle in GFQT as follows. Let $\mathcal{A}$ be a Lie algebra over F_p which is treated as a symmetry algebra, then the particle is

elementary if the set of its states forms an irreducible representation (IR) of $\mathcal{A}$ in $F(p^n)$. Representations of Lie algebras in spaces with nonzero characteristic are called modular representations. There exists a well developed theory of such representations. One of the well known results is the Zassenhaus theorem [12] that any modular IR is finite dimensional.

As argued in Refs. [8, 9], standard theories based on de Sitter (dS) algebra so(1,4) or anti de Sitter (AdS) algebra so(2,3) can be generalized to theories based on a Galois field while there are problems with the generalization of the theory based on Poincare algebra. The reasons are the following. It is clear that in theories based on Galois fields there can be no dimensional quantities and all physical quantities are discrete. In standard dS or AdS invariant theories all physical quantities are dimensionless and discrete in units $\hbar/2 = c = 1$ while in Poincare invariant theory the energy and momentum necessarily have a continuous spectrum. From the formal point of view, the representation operators of the Poincare algebra can also be chosen dimensionless, e.g. in Planck units. In Poincare invariant theories over a Galois field one has to choose a quantum of length. If this quantum is the Planck distance then the quantum of mass will be the Planck mass, which is much greater than the masses of elementary particles.

The existing astronomical data indicate that the cosmological constant is small and positive rather than negative. This is an argument in favor of so(1,4) vs. so(2,3). On the other hand, in QFT and its generalizations (string theory, M-theory etc.) a theory based on so(1,4) encounters serious difficulties and the choice of so(2,3) is preferable (see e.g. Ref. [13]). IRs of the so(2,3) algebra have a lot in common with IRs of the Poincare algebra. In particular, in IRs of the so(2,3) algebra the AdS Hamiltonian is either strictly positive or strictly negative and the supersymmetric generalization is possible. For these reasons in the present paper, for illustration of what happens when complex numbers are replaced by a Galois field, we assume that $\mathcal{A}$ is the modular analog of the algebra so(2,3).

In Sects. 2 and 3 we explicitly construct modular IRs describing elementary particles in GFQT. The calculations are straightforward and do not require any specific technique, so all the results can be reproduced even by readers who previously have not had any practice in calculations with Galois fields.

2 Modular IRs of the sp(2) algebra

The key role in constructing modular IRs of the so(2,3) algebra is played by modular IRs of the sp(2) subalgebra. They are described by a set of operators $(a', a", h)$ satisfying the commutation relations

$$[h, a'] = -2a' \quad [h, a"] = 2a" \quad [a', a"] = h \tag{1}$$

The Casimir operator of the second order for the algebra (1) has the form

$$K = h^2 - 2h - 4a"a' = h^2 + 2h - 4a'a" \tag{2}$$

We first consider representations with the vector e_0 such that

$$a'e_0 = 0, \quad he_0 = q_0 e_0 \tag{3}$$

where $q_0 \in F_p$ and $f(q_0) > 0$. Recall that we consider the representation in a linear space over F_{p^k} where k is a natural number. Denote $e_n = (a")^n e_0$. Then it follows from Eqs. (2) and (3), that

$$he_n = (q_0 + 2n)e_n, \quad Ke_n = q_0(q_0 - 2)e_n, \tag{4}$$

$$a'a"e_n = (n+1)(q_0 + n)e_n \tag{5}$$

One can consider analogous representations in the standard theory. Then q_0 is a positive real number, $n = 0, 1, 2, ...\infty$ and the elements e_n form a basis of the IR. In this case e_0 is a vector with a minimum eigenvalue of the operator h (minimum weight) and there are no vectors with the maximum weight. The operator h is positive definite and bounded below by the quantity q_0. For these reasons the above modular IRs can be treated as modular analogs of such standard IRs that h is positive definite.

Analogously, one can construct modular IRs starting from the element e'_0 such that

$$a"e'_0 = 0, \quad he'_0 = -q_0 e'_0 \tag{6}$$

and the elements e'_n can be defined as $e'_n = (a')^n e'_0$. Such modular IRs are analogs of standard IRs where h is negative definite while in the modular case Eqs. (3) and (6) define the same IRs. This is clear from the following consideration.

The set $(e_0, e_1, ...e_N)$ will be a basis of IR if $a"e_i \neq 0$ for $i < N$ and $a"e_N = 0$. These conditions must be compatible with $a'a"e_N = 0$. As follows from Eq. (5), N is defined by the condition $q_0 + N = 0$ in F_p. As a result, if q_0 is one of the numbers $1, ...p-1$ then $N = p - q_0$ and the dimension of IR is equal to $p - q_0 + 1$ (in agreement with the Zassenhaus theorem [12]). It is easy to see that e_N satisfies Eq. (6) and therefore it can be identified with e'_0.

Let us forget for a moment that the eigenvalues of the operator h belong to F_p and will treat them as integers. Then, as follows from Eq. (4), the eigenvalues are

$$q_0, q_0 + 2, ..., 2p - 2 - q_0, 2p - q_0.$$

Therefore, if $f(q_0) > 0$ and $f(q_0) \ll p$, the maximum eigenvalue equals $2p - q_0$, i.e. it is of order $2p$.

3 Modular IRs of the so(2,3) algebra

Standard IRs of the AdS so(2,3) algebra relevant for describing elementary particles have been considered by many authors. The description in this section is a combination of two elegant ones given in Ref. [14] for standard IRs and Ref. [15] for modular IRs. In the standard theory the representation operators of the so(2,3) algebra in units $\hbar/2 = c = 1$ are given by

$$[M^{ab}, M^{cd}] = -2i(g^{ac}M^{bd} + g^{bd}M^{cd} - g^{ad}M^{bc} - g^{bc}M^{ad}) \tag{7}$$

where a, b, c, d take the values 0,1,2,3,5 and $M^{ab} = -M^{ba}$. The diagonal metric tensor has the components $g^{00} = g^{55} = -g^{11} = -g^{22} = -g^{33} = 1$. In these units the spin of fermions is odd, and the spin of bosons is even. If s is the particle spin then the corresponding IR of the su(2) algebra has the dimension $s + 1$.

Note that our definition of the AdS symmetry on quantum level does not involve the cosmological constant. It appears only if one is interested in interpreting results in terms of the de Sitter spacetime or in the Poincare limit. If one assumes that spacetime is fundamental, then in the spirit of General Relativity it is natural to think that the empty space is flat, i.e. that the cosmological constant equals zero. This was the subject of the well-known dispute between Einstein and de Sitter described in the literature. In QFT the cosmological constant is given by a contribution of vacuum diagrams, and the problem is to explain why it is so small. On the other hand, if we assume that symmetry on quantum level in our formulation is more fundamental, then the cosmological constant problem does not arise at all. Instead, we have a problem of why nowadays Poincare symmetry is such a good approximate symmetry. This is a problem of cosmology rather then quantum physics.

If a modular IR is considered in a linear space over F_{p^2} with $p = 3$ $(mod\ 4)$ then Eq. (7) is also valid but in the general case it is convenient to work with another set of ten operators. Let $(a'_j, a_j", h_j)$ $(j = 1, 2)$ be two independent sets of operators satisfying the commutation relations for the sp(2) algebra

$$[h_j, a'_j] = -2a'_j \quad [h_j, a_j"] = 2a_j" \quad [a'_j, a_j"] = h_j \tag{8}$$

The sets are independent in the sense that for different j they mutually commute with each other. We denote additional four operators as $b', b", L_+, L_-$. The operators $L_3 = h_1 - h_2, L_+, L_-$ satisfy the commutation relations of the su(2) algebra

$$[L_3, L_+] = 2L_+ \quad [L_3, L_-] = -2L_- \quad [L_+, L_-] = L_3 \tag{9}$$

while the other commutation relations are

$$[a'_1, b'] = [a'_2, b'] = [a_1", b"] = [a_2", b"] = [a'_1, L_-] = [a_1", L_+] =$$
$$[a'_2, L_+] = [a_2", L_-] = 0 \quad [h_j, b'] = -b' \quad [h_j, b"] = b" \quad [h_1, L_\pm] = \pm L_\pm$$
$$[h_2, L_\pm] = \mp L_\pm \quad [b', b"] = h_1 + h_2 \quad [b', L_-] = 2a'_1 \quad [b', L_+] = 2a'_2$$
$$[b", L_-] = -2a_2" \quad [b", L_+] = -2a_1" \quad [a'_1, b"] = [b', a_2"] = L_-$$
$$[a'_2, b"] = [b', a_1"] = L_+ \quad [a'_1, L_+] = [a'_2, L_-] = b'$$
$$[a_2", L_+] = [a_1", L_-] = -b" \tag{10}$$

At first glance these relations might seem rather chaotic but in fact they are very natural in the Weyl basis of the so(2,3) algebra.

In spaces over F_{p^2} with $p = 3\ (mod\ 4)$ the relation between the above sets of ten operators is

$$M_{10} = i(a_1" - a'_1 - a_2" + a'_2) \quad M_{15} = a_2" + a'_2 - a_1" - a'_1$$
$$M_{20} = a_1" + a_2" + a'_1 + a'_2 \quad M_{25} = i(a_1" + a_2" - a'_1 - a'_2)$$
$$M_{12} = L_3 \quad M_{23} = L_+ + L_- \quad M_{31} = -i(L_+ - L_-)$$
$$M_{05} = h_1 + h_2 \quad M_{35} = b' + b" \quad M_{30} = -i(b" - b') \tag{11}$$

which is why the sets are equivalent. The relations (8-10) are more general since they can be used when the representation space is a space over F_{p^k} where k is arbitrary.

We use the basis in which the operators (h_j, K_j) $(j = 1, 2)$ are diagonal. Here K_j is the Casimir operator (2) for algebra $(a'_j, a_j", h_j)$. For constructing IRs we need operators relating different representations of the sp(2)×sp(2) algebra. By analogy with Refs. [14, 15], one of the possible choices is

$$A^{++} = b"(h_1 - 1)(h_2 - 1) - a_1"L_-(h_2 - 1) - a_2"L_+(h_1 - 1) + a_1"a_2"b'$$
$$A^{+-} = L_+(h_1 - 1) - a_1"b' \quad A^{-+} = L_-(h_2 - 1) - a_2"b' \quad A^{--} = b' \tag{12}$$

We consider the action of these operators only on the space of 'minimal' sp(2)×sp(2) vectors, i.e. such vectors x that $a'_j x = 0$ for $j = 1, 2$, and x is the eigenvector of the operators h_j. If x is a minimal vector such that $h_j x = \alpha_j x$ then $A^{++}x$ is the minimal eigenvector of the operators h_j with the eigenvalues $\alpha_j + 1$, $A^{+-}x$ - with the eigenvalues $(\alpha_1 + 1, \alpha_2 - 1)$, $A^{-+}x$ - with the eigenvalues $(\alpha_1 - 1, \alpha_2 + 1)$, and $A^{--}x$ - with the eigenvalues $\alpha_j - 1$.

By analogy with Refs. [14, 15], we require the existence of the vector e_0 satisfying the conditions

$$a'_j e_0 = b' e_0 = L_+ e_0 = 0 \quad h_j e_0 = q_j e_0 \quad (j = 1, 2) \tag{13}$$

where $q_j \in F_p$, $f(q_j) > 0$ and $f(q_1 - q_2) \geq 0$. It is well known (see e.g. Ref. [9]) that $M^{05} = h_1 + h_2$ is the AdS analog of the energy operator. As follows from Eqs. (8)

and (10), the operators (a_1', a_2', b') reduce the AdS energy by two units. Thus e_0 is an analog of the state with the minimum energy which can be called the rest state, and the spin in our units is equal to the eigenvalue of the operator $L_3 = h_1 - h_2$ in that state. For these reasons we use s to denote $q_1 - q_2$ and m to denote $q_1 + q_2$. In the standard classification [14], the massive case is characterized by the condition $q_2 > 1$ and the massless case — by the condition $q_2 = 1$. There also exist two exceptional IRs discovered by Dirac [16] (Dirac singletons). As shown in Ref. [8], the modular analog of Dirac singletons is simple and the massless case has been discussed in details in Ref. [17]. For these reasons in the present paper we consider only the massive case.

As follows from the above remarks, the elements

$$e_{nk} = (A^{++})^n (A^{-+})^k e_0 \tag{14}$$

represent the minimal sp(2)×sp(2) vectors with the eigenvalues of the operators h_1 and h_2 equal to $Q_1(n,k) = q_1 + n - k$ and $Q_2(n,k) = q_2 + n + k$, respectively. It can be shown by a direct calculation that

$$A^{--}A^{++}e_{nk} = (n+1)(m+n-2)(q_1+n)(q_2+n-1)e_{nk} \tag{15}$$

$$A^{+-}A^{-+}e_{nk} = (k+1)(s-k)(q_1-k-2)(q_2+k-1)e_{nk} \tag{16}$$

As follows from these expressions, in the massive case k can assume only the values $0, 1, ...s$ and in the standard theory $n = 0, 1, ...\infty$. In the modular case $n = 0, 1, ...n_{max}$ where n_{max} is the first number for which the r.h.s. of Eqs. (15) becomes zero in F_p, consequently $n_{max} = p + 2 - m$.

The full basis of the representation space can be chosen in the form

$$e(n_1 n_2 nk) = (a_1")^{n_1} (a_2")^{n_2} e_{nk} \tag{17}$$

In the standard theory n_1 and n_2 can be any natural numbers. However, as follows from the results of the preceding section, Eq. (8) and the properties of the A operators,

$$\begin{gathered} n_1 = 0, 1, ...N_1(n,k) \quad n_2 = 0, 1, ...N_2(n,k) \\ N_1(n,k) = p - q_1 - n + k \quad N_2(n,k) = p - q_2 - n - k \end{gathered} \tag{18}$$

As a consequence, the representation is finite dimensional in agreement with the Zassenhaus theorem [12]. Moreover, it is finite since any Galois field is finite.

In standard Poincare and AdS theories there also exist IRs with negative energies. They can be constructed by analogy with positive energy IRs. Instead of Eq. (13) one can require the existence of the vector e_0' such that

$$a_j" e_0' = b" e_0' = L_- e_0' = 0 \quad h_j e_0' = -q_j e_0' \quad (e_0', e_0') \neq 0 \quad (j = 1, 2) \tag{19}$$

where the quantities q_1, q_2 are the same as they are for positive energy IRs. It is obvious that positive and negative energy IRs are fully independent since the spectrum

of the operator M^{05} for such IRs is positive and negative, respectively. However, *the modular analog of a positive energy IR characterized by q_1, q_2 in Eq. (13), and the modular analog of a negative energy IR characterized by the same values of q_1, q_2 in Eq. (19) represent the same modular IR.* This is the crucial difference between the standard quantum theory and GFQT, and a proof is given below.

Let e_0 be a vector satisfying Eq. (13). Denote $N_1 = p - q_1$ and $N_2 = p - q_2$. Our goal is to prove that the vector $x = (a_1")^{N_1}(a_2")^{N_2}e_0$ satisfies the conditions (19), i.e. x can be identified with e'_0.

As follows from the definition of N_1, N_2, the vector x is the eigenvector of the operators h_1 and h_2 with the eigenvalues $-q_1$ and $-q_2$, respectively, and in addition it satisfies the conditions $a_1"x = a_2"x = 0$. Let us prove that $b"x = 0$. Since $b"$ commutes with the $a_j"$, we can write $b"x$ in the form

$$b"x = (a_1")^{N_1}(a_2")^{N_2}b"e_0 \tag{20}$$

As follows from Eqs. (10) and (13), $a'_2 b"e_0 = L_+e_0 = 0$ and $b"e_0$ is the eigenvector of the operator h_2 with the eigenvalue $q_2 + 1$. Thus, $b"e_0$ is the minimal vector of the sp(2) IR which has the dimension $p - q_2 = N_2$. Therefore $(a_2")^{N_2}b"e_0 = 0$ and $b"x = 0$.

The next step is to show that $L_-x = 0$. As follows from Eq. (10) and the definition of x,

$$L_-x = (a_1")^{N_1}(a_2")^{N_2}L_-e_0 - N_1(a_1")^{N_1-1}(a_2")^{N_2}b"e_0 \tag{21}$$

We have already shown that $(a_2")^{N_2}b"e_0 = 0$, and hence it suffices to prove that the first term in the r.h.s. of Eq. (21) equals zero. As follows from Eqs. (10) and (13), $a'_2 L_-e_0 = b'e_0 = 0$ and L_-e_0 is the eigenvector of the operator h_2 with the eigenvalue $q_2 + 1$. Thus, $(a_2")^{N_2}L_-e_0 = 0$ and the proof is completed.

Let us assume for a moment that the eigenvalues of the operators h_1 and h_2 are treated not as elements of F_p but as integers. As follows from the consideration in the preceding section, if $|f(q_j)| \ll p$ then one modular IR of the so(2,3) algebra corresponds to a standard positive energy IR in the region where the energy is positive and much less than p. At the same time, it corresponds to an IR with the negative energy in the region where the AdS energy is close to $4p$ but less than $4p$.

4 Discussion

In the present paper we discuss the description of free elementary particles in a quantum theory based on a Galois field (GFQT). As noted in Sect. 1, GFQT does not contain infinities at all and all operators are automatically well defined. In my discussions with physicists, some of them commented that this is an approach where a cutoff (the characteristic p of the Galois field) is introduced from the beginning and for this reason there is nothing unusual in the fact that the theory does not have

infinities. It has a large number p instead, and this number can be practically treated as infinite. In the spirit of such ideas, my original motivation for investigating GFQT was as follows. Let us take standard QED in dS or AdS space, write the Hamiltonian and other operators in angular momentum basis and replace standard IRs for the electron, positron and photon by corresponding modular IRs. One might treat this as an attempt to substantiate standard momentum regularizations (e.g. the Pauli-Villars regularization) at momenta p/R, where R is the radius of the Universe. In other terms this might be treated as introducing fundamental length of order R/p. The results of the present paper give a clear explanation why such a naive attempt fails. Before discussing the results in greater details, we first recall how elementary particles are described in standard quantum theory.

Unitary IRs of the Poincare or AdS groups (or IRs of the Poincare and AdS algebras by Hermitian operators) used for description of elementary particles have the property that for each IR, the Poincare or AdS Hamiltonian is either positive definite or negative definite. In the first case, the energy has the spectrum in the range $[mass, \infty)$, while in the second case it has the spectrum in the range $(-\infty, -mass]$. The standard requirement is that the energy cannot be negative, but it is not a matter of principle but a matter of convention. For example, in Poincare invariant theory, one might define the energy not as $\sqrt{m^2 + \mathbf{p}^2}$ but as $-\sqrt{m^2 + \mathbf{p}^2}$. Then the theory will not contradict the energy conservation if the sign of energy is the same for all particles and one might conclude that the existence of IRs with both signs of the energy is redundant. Such a conclusion is premature and one of the reasons is described below.

In QFT a local description of spin 1/2 particles is possible only in the framework of the Dirac equation, which combines two IRs with the same mass and opposite energy signs. IRs with the positive energy are usually associated with a particle and then the second IR is associated with its antiparticle. The problem of negative energies is then solved by quantization, after which the energies of both, the particle and its antiparticle become positive definite.

Let us now discuss what happens in GFQT. We start from the rest state where energy=mass and the value of the energy is "positive", i.e. is in the right half-plane in Fig. 1, which is why one might treat this construction as a modular analog of the standard representation with the minimum weight. When we gradually construct states with "higher and higher" energies then in contrast with the standard case we are moving not in the positive direction of the x axis, but along the circumference counterclockwise. Sooner or later we will arrive at the left half-plane in Fig. 1 (i.e. to states with "negative energies") and finally we will arrive to the state where energy=-mass. In mathematical terminology this means that a modular analog of IR with the minimum weight is simultaneously a modular analog of IR with the maximum weight, while from the point of view of physics, one modular IR describes a particle and its antiparticle simultaneously.

In standard quantum theory the existence of antiparticles follows from the

assumption that a full set of quantum states can be obtained by quantizing a local covariant equation, e.g. the Dirac equation. A question arises that if locality is only an approximation then it is not clear whether the notion of antiparticles is exact or approximate. At the same time, the above construction shows that *in GFQT the existence of antiparticles follows from the fact that any Galois field is finite.*

If p is treated only as a cutoff parameter then one might expect that in GFQT a particle is described by IR where the energy is in the range from m to a value of order p and an antiparticle is described by IR where the energy is in the range from a value of order $-p$ to $-m$. However, since in Galois fields the rules of arithmetic are different, such an expectation is not correct.

In QFT the fact that a particle and its antiparticle have the same masses and spins follows from the CPT theorem, which is a consequence of locality. One can again pose a question of what happens if locality is only an approximation: in that case the equality of masses and spins is exact or approximate? Consider a simple model when electromagnetic and weak interactions are absent, then the fact that the proton and the neutron have the same masses and spins is irrelevant of locality or nonlocality; it is only a consequence of the fact that the proton and the neutron belong to the same isotopic multiplet. In other words, they are simply different states of the same object - the nucleon. We see that in GFQT the situation is analogous. The fact that a particle and its antiparticle have the same masses and spins is irrelevant of locality or nonlocality and is simply a consequence of the fact that they are different states of the same object since they belong to the same IR. Note also, that as shown in Ref. [18], even in *standard* theory with the invariance group SO(1,4) the only possible interpretation of IRs is that they describe a particle and its antiparticle simultaneously.

In summary, while in standard theory the existence of antiparticles depends on additional assumptions, in GFQT it is inevitable. Therefore, *the very existence of antiparticles is a strong indication that nature is described by a finite field rather than by complex numbers.*

Another reason why the naive attempt described in the beginning of this section fails is that in GFQT there can be no neutral elementary particles. In particular neither the photon nor the graviton (if it exists) can be elementary. This can be proved by quantizing IRs and the results will be described elsewhere.

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